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A geometric classification of Lagrangian functions and the reduction of evolution space

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Abstract. A geometric classification of degenerate time-dependent Lagrangian systems is given and the reduction of evolution space is analysed. General properties of semiregular Lagrangians (type II) are discussed and particular attention is paid to the reduction of completely degenerate Lagrangians (type III) which are considered in detail.

1. Introduction

Degenerate Lagrangians are ubiquitous in physics and extensive literature has been devoted to discussing them from several points of view and for several purposes. The study of the reduction of evolution space of an autonomous degenerate Lagrangian was initiated in a previous paper [4]. The reduction process involves quotienting out the gauge degrees of freedom associated with the kernel of the Cartan 2-form defined by the Lagrangian function, and with studying the existence, or non-existence, of a non-degenerate Lagrangian system on the quotient space. Apart from some technical restrictions on the family of degenerate Lagrangians suitable for such study (the most restrictive from the physical point of view is the non-allowance of secondary constraints) it was proved that only a special class of these (called type II) are good candidates for the purposes of reduction. One of the most interesting by-products of this program was to make explicit the existence of non-trivial constraints on the dimension of the gauge algebra (the Lie algebra of vectors lying in the kernel of the Cartan form). It happens that type II Lagrangians are the most similar to regular Lagrangians; for instance they are the non-regular Lagrangians having the highest-rank Cartan form and they always possess a second-order differential equation (SODE).

The program started in [4] was unfinished, not only because generic degenerate Lagrangians (those of type III) were not considered, but because the discussion was concentrated in the autonomous case. Recently, new ideas have been introduced in the discussion of time-dependent degenerate Lagrangians [2] showing that new aspects arise in this context that extend in a non-trivial way the results in [4].

In this paper we will revisit the program of reduction of degenerate Lagrangians considering from the beginning the non-autonomous case, and concentrating our attention on the generic case, i.e. not only in the semiregular situation. First of all we will describe the classification of non-autonomous Lagrangians extending the classification in [4]. We will investigate in general the process of reduction of evolution space as a second step and, afterwards, we will study in more detail the reduction

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of type II Lagrangians and special types of type III Lagrangians, the completely degenerate and those having a gauge distribution of almost tangent type.

The organization of the paper is as follows. Sections 2 and 3 are devoted to reviewing general notions on the geometry of evolution space and to establishing the classification of degenerate non-autonomous Lagrangians systems. In section 4 we describe the general reduction program for degenerate non-autonomous Lagrangian systems and in particular we will discuss type II and several families of type III Lagrangians in subsections 4.1 and 4.2.

2. Geometric structures on evolution space

2.1. Some natural tensor fields on $J^1(\mathbb{R}, Q)$

Let Q be a manifold of dimension m, the configuration space of a dynamical system with local coordinates q^i, and TQ the tangent bundle of Q with natural coordinates $q^i, \dot{q}^i$. The evolution space for time-dependent mechanical systems is the first jet bundle of smooth maps from $\mathbb{R}$ to Q, denoted by $J^1(\mathbb{R}, Q)$ and, as is well known, there exists a canonical isomorphism between $J^1(\mathbb{R}, Q)$ and $TQ \times \mathbb{R}$. Local coordinates in evolution space are simply $t, q^i, \dot{q}^i$. Therefore in what follows, when we want to refer to the evolution space, we will write $TQ \times \mathbb{R}$.

Let $\tilde{\tau}_Q : TQ \times \mathbb{R} \to Q \times \mathbb{R}$ be the projection induced by the canonical projection $\tau_Q : TQ \to Q$ and the identity on $\mathbb{R}$. Then $J^1(\mathbb{R}, Q)$ becomes in this way a vector bundle over $Q \times \mathbb{R}$ with projection map $\tilde{\tau}_Q$ and fibres the tangent spaces $T_q Q$. We define the vertical bundle $V(TQ \times \mathbb{R})$ as the sub-bundle $\ker \tilde{\tau}_{Q*}$ of $T(TQ \times \mathbb{R})$, i.e. the set of vectors $V \in T(TQ \times \mathbb{R})$ such that $\tilde{\tau}_{Q*}(V) = 0$. Locally, a vertical vector field has the form $V = V^i(q, \dot{q}, t)\partial/\partial\dot{q}^i$.

There are several geometrical structures on the evolution space, some arising naturally in $TQ \times \mathbb{R}$ and some associated with a given function L. Most of them are related to the canonical, integrable, almost tangent structure on TQ, the vertical endomorphism S, a rank-m $(1,1)$ tensor field on TQ such that $\ker S = \operatorname{Im} S$ and whose Nijenhuis tensor N_S vanishes. In natural coordinates, the local expression for S is given by

$$S = \frac{\partial}{\partial \dot{q}^i} \otimes \mathrm{d}q^i . \tag{1}$$

The other main geometrical ingredient of TQ is the Liouville vector field Δ which is the infinitesimal generator of the dilations along the fibres on TQ and is locally expressed by $\Delta = \dot{q}^i\partial/\partial\dot{q}^i$. In a precise sense S and Δ are the main geometrical structures on TQ because under certain conditions they characterize the tangent bundle TQ (see [6] and [8]).

On the other hand, in $TQ \times \mathbb{R}$ there exists, apart from the Liouville vector field Δ, a canonical tensor field of type $(1,1)$ given by (see [5] for more details on the geometry of $J^1(\mathbb{R}, Q)$)

$$\tilde{S} = S - \Delta \otimes \mathrm{d}t . \tag{2}$$

We can observe that rank $\tilde{S} = m$, $\tilde{S}^2 = 0$ but it is not integrable, $N_{\tilde{S}} = -S \otimes \mathrm{d}t \neq 0$. Locally, using the family of local-contact 1-forms $\theta^i = \mathrm{d}q^i - \dot{q}^i \mathrm{d}t$, we can write $\tilde{S} = \partial/\partial \dot{q}^i \otimes \theta^i$.

Alternatively, we can introduce a different sort of structure, an almost s-tangent structure [10]. Let us recall that an almost s-tangent structure on a $(2m+1)$-dimensional manifold M is a triple $(\bar{S}, \tau, \gamma)$, where $\bar{S}$ is a tensor field of type $(1,1)$, τ is a 1-form and γ a vector field on M such that (i) $i_\gamma \tau = 1$, (ii) $\bar{S}^2 = \gamma \otimes \tau$, and (iii) rank $\bar{S} = m+1$. An almost s-tangent structure is called integrable if $N_{\bar{S}} = 0$ and τ is closed. It is obvious that the triple $(\bar{S}, \mathrm{d}t, \partial/\partial t)$ with

$$\bar{S} = S + \frac{\partial}{\partial t} \otimes \mathrm{d}t \tag{3}$$

defines an almost s-tangent structure on $TQ \times \mathbb{R}$. It is possible to show that an integrable almost s-tangent structure characterizes $TQ \times \mathbb{R}$ [3].

There are several invariant definitions of second-order differential equations in evolution space. A vector field Γ in the evolution space $TQ \times \mathbb{R}$ is called a second-order differential equation (SODE for short), if $S(\Gamma) = \Delta$ and $i_\Gamma \mathrm{d}t = 1$. Clearly the local expression for a SODE is

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + f^j \frac{\partial}{\partial \dot{q}^j} \tag{4}$$

and it is clear that Γ is a SODE if $\tilde{S}(\Gamma) = 0$ and $S(\Gamma) = \Delta$.

We will collect all the information on the structure of the kernel and the image of the different (1, 1) tensor fields $S, \tilde{S}$ and $\bar{S}$ defined so far in the following lemma.

Lemma 1. With the definitions above, we have
(i) $\ker S = V(TQ \times \mathbb{R}) \oplus T\mathbb{R}$
(ii) $\mathrm{Im}\, S = V(TQ \times \mathbb{R})$
(iii) $\ker \bar{S} = V(TQ \times \mathbb{R})$
(iv) $\mathrm{Im}\, \bar{S} = V(TQ \times \mathbb{R}) \oplus T\mathbb{R}$
(v) $\mathrm{Im}\, \tilde{S} = V(TQ \times \mathbb{R})$
(vi) $\tilde{S}$ is zero over vertical vector fields and SODEs.

2.2. *Lagrangian systems*

Let $L : TQ \times \mathbb{R} \to \mathbb{R}$ be a smooth function. The Cartan 1-form associated with L is given by

$$\Theta_L = L\mathrm{d}t + \mathrm{d}L \circ \tilde{S} \tag{5}$$

and the Cartan 2-form associated with L is the exterior derivative of Θ_L, $\Omega_L = \mathrm{d}\Theta_L$. The 2-form Ω_L together with $\mathrm{d}t$ will sometimes define a cosymplectic structure [11] on the evolution space $TQ \times \mathbb{R}$. In fact, a cosymplectic structure (or an almost-contact structure) on a $(2m+1)$-dimensional manifold M is a triple (M, Ω, η), such that Ω is a closed 2-form, η is a closed 1-form and $\Omega^m \wedge \eta \neq 0$. In particular, we can observe that $\Omega^m \wedge \eta$ defines a volume form on M, and that Ω is necessarily of maximal rank $2m$. Because of this there exists a unique vector field Γ on M such that $i_\Gamma \Omega = 0$ and $i_\Gamma \eta = 1$, called the Reeb field of the cosymplectic structure. It is possible to relax the maximal rank condition of a cosymplectic structure and we obtain the notion of a precosymplectic structure on M as a triple (M, Ω, η) such that Ω is a closed 2-form, η is a closed 1-form on M and $\Omega^r \wedge \eta \neq 0$, $\Omega^{r+1} = 0$. Therefore Ω has constant rank $2r$ ($r < m$). It is clear that the distribution $\ker \Omega \cap \ker \eta$ is involutive.

If Ω_L is of maximal rank $2m$, then we will say that L is a regular Lagrangian and then, $(\Omega_L, \mathrm{d}t)$ defines a cosymplectic structure on $TQ \times \mathbb{R}$. Therefore there exists a unique vector field Γ on $TQ \times \mathbb{R}$ such that

$$i_\Gamma \Omega_L = 0 \qquad i_\Gamma \mathrm{d}t = 1\,. \tag{6}$$

The vector field Γ is called the Euler–Lagrange vector field of L and it can be shown that it is a SODE; its integral curves are the solutions of Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \qquad \dot{q}^i = \frac{\mathrm{d}q^i}{\mathrm{d}t} \qquad 1 \leqslant i \leqslant m\,.$$

In the following sections we are going to discuss the situation when Ω_L is not of maximal rank. In such cases there will be problems both with the uniqueness and globality of the solutions. In order to restrict as much as possible some of the technical difficulties involved in dealing with the global existence of vector fields satisfying (6) we will adopt several restrictions that we will describe immediately.

3. A classification of Lagrangians

In this section we are going to extend to the time-dependent case the classification of Lagrangians introduced in [4]. We will get a similar scheme; the dimensions of the kernel of the precosymplectic structure and its vertical part cannot take arbitrary values. As in the autonomous case, in order to avoid the difficulties involved in considering quotient spaces in the process of reduction we will first assume that the pair $(\Omega_L, \mathrm{d}t)$ define a precosymplectic structure on $TQ \times \mathbb{R}$. In that case we know that the distribution K given by $\ker \Omega_L \cap \ker \mathrm{d}t$ is involutive. This distribution will be called the *gauge distribution* of the degenerate Lagrangian L. Secondly we will also require that the foliation defined by the gauge distribution K is a fibration. This implies that the quotient space $TQ \times \mathbb{R}/K$ admits a manifold structure, and the projection $\pi: TQ \times \mathbb{R} \to TQ \times \mathbb{R}/K$ becomes a surjective submersion. Finally, it is possible to show that if $(\Omega_L, \mathrm{d}t)$ is precosymplectic, L admits global dynamics (see for instance [2]), i.e. there exists at least one vector field X satisfying equations (6).

Summarizing, we adopt the following basic technical assumptions on the function L that can be considered in what follows as the definition of a Lagrangian function:

(A1) $(\Omega_L, \mathrm{d}t)$ is precosymplectic;

(A2) The foliation defined by K is a fibration.

Remarks. A few remarks on conditions (A1), (A2) are in order here.

(i) There are no fundamental reasons (apart from technical reasons) for the assumptions (A1), (A2). It is easy to construct functions L such that Ω_L does not satisfy (A1), (A2). On the other hand almost all relevant physical examples satisfy (A2).

(ii) The existence of global dynamics implies that there are no secondary or higher-order constraints for the Lagrangian L. Obviously there are many important situations where we face higher-order constraints. Unfortunately the geometrical structure of the constraint algorithm is completely different to the geometric structures described so far to reduce degenerate Lagrangians. To make both processes compatible requires further refinements of the theory.

(iii) It is immediate that if L admits global dynamics, then the general solution of the dynamical equations will have the form $\tilde{\Gamma} = \Gamma + X$, where Γ is a particular solution and X is a vector field in K. However, in general there is no solution $\tilde{\Gamma}$ of the dynamical equations (6) which is a SODE. We will show later that for some special Lagrangians (type II) there is always a $\tilde{\Gamma}$ which is a SODE.

Now, we are going to give a classification of Lagrangian systems. The first important result relates $\tilde{S}$ and Ω_L.

Lemma 2. If $L : TQ \times \mathbb{R} \to \mathbb{R}$ is a Lagrangian function, then

$$i_{\tilde{S}}\Omega_L = 0\,. \tag{7}$$

Proof. Simple computations show that the only term of $i_{\tilde{S}}\Omega_L$, in local coordinates $t, q^i, \dot{q}^i$, that does not vanish trivially is

$$i_{\tilde{S}}\Omega_L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial q^i}\right).$$

The computation of Ω_L shows that the coefficient of $\mathrm{d}t \wedge \mathrm{d}\dot{q}^i$ is $\dot{q}^j(\partial^2 L/\partial\dot{q}^j\partial\dot{q}^i)$. On the other hand, the coefficient of $\mathrm{d}\dot{q}^i \wedge \mathrm{d}q^i$ is $\partial^2 L/\partial\dot{q}^j\partial\dot{q}^i$. Collecting these results and taking into account $i_{\tilde{S}}\Omega_L(X, Y) = \Omega_L(\tilde{S}(X), Y) + \Omega_L(X, \tilde{S}(Y))$ we get the desired result. □

It is important to notice that $i_S\Omega_L \neq 0$ and $i_{\bar{S}}\Omega_L \neq 0$. From equation (7) and lemma 1 we easily get

$$\tilde{S}(\ker\Omega_L) \subset \ker\Omega_L \cap V(TQ \times \mathbb{R}) \tag{8}$$

and from the definitions of the respective tensors we get

$$S\,|_{\ker \mathrm{d}t} = \tilde{S}\,|_{\ker \mathrm{d}t} = \bar{S}\,|_{\ker \mathrm{d}t}\,. \tag{9}$$

Then it is clear that

Lemma 3. Let L be a Lagrangian function and K its gauge distribution. Then

$$S(K) = \tilde{S}(K) = \bar{S}(K) \subset K\,. \tag{10}$$

Proof. Because of (9) the tensors S, $\tilde{S}$ and $\bar{S}$ agree when restricted to $\ker \mathrm{d}t$, so they will agree on $K = \ker \mathrm{d}t \cap \ker\Omega_L$. But $\tilde{S}(K) \subset \tilde{S}(\ker\Omega_L)$ which is contained in $\ker\Omega_L \cap V(TQ \times \mathbb{R})$ by (8), then the result follows from the inclusion $\ker\Omega_L \cap V(TQ \times \mathbb{R}) \subset K$. □

It is an immediate consequence of this that

Lemma 4. In the conditions above

$$\ker(\tilde{S}\,|_K) = K \cap V(TQ \times \mathbb{R})\,. \tag{11}$$

Proof. From lemma 3 we get $\tilde{S}(K) \subset K$ and from (9) we conclude that $\bar{S}\,|_K = \tilde{S}\,|_K$. Then because of lemma 1 we get the result. □

Finally, we can conclude this argument with the following theorem.

Theorem 1. If L is a Lagrangian function and we denote by $V(K)$ the vertical part of its gauge distribution K, $V(K) = K \cap V(TQ \times \mathbb{R})$, then

$$\dim V(K) \leqslant \dim K \leqslant 2\dim V(K)\,. \tag{12}$$

Proof. Because of (8) and (10) it is clear that $\tilde{S}(K) \subset K \cap V(TQ \times \mathbb{R}) = V(K)$. But because of lemma 4 we have $V(K) = \ker(\tilde{S}\,|_K)$. Then $\dim K = \dim \tilde{S}(K) + \dim\ker(\tilde{S}\,|_K) = \dim \tilde{S}(K) + \dim V(K) \leqslant 2\dim V(K) \Rightarrow \dim V(K) \geqslant \frac{1}{2}\dim K$.

Using this property we can already distinguish three types of Lagrangians extending the classification in [4]:

(i) Type I: if $\dim K = \dim V(K) = 0$;
(ii) Type II: if $\dim K = 2\dim V(K) \neq 0$;
(iii) Type III: if $\dim K < 2\dim V(K)$.

Type I Lagrangians are just regular Lagrangians. Type II Lagrangians have very interesting properties from the point of view of reduction and a class of them (see section 4.1) have been already studied by de León *et al* [2].

Remarks.

(i) If we are discussing time-independent Lagrangians, it is easy to show that $K = \ker\omega_L$, where ω_L is the Cartan 2-form of the autonomous Lagrangian L. Then this classification reduces to the classification given in [4].

(ii) It is easy to show that

$$V(\ker\Omega_L) = \ker\Omega_L \cap \operatorname{Im} S = K \cap \operatorname{Im} S = V(K)\,.$$

Then because of (12), we get

$$\dim K \leqslant 2\dim V(\ker\Omega_L)$$

and taking into account $\dim\ker\Omega_L \leqslant \dim K + 1$, we have that

$$\dim\ker\Omega_L \leqslant 2\dim V(\ker\Omega_L) + 1\,.$$

Therefore, another equivalent characterization for the different types of Lagrangians above is:

(i) Type I (regular Lagrangians): $\dim\ker\Omega_L = 1$ and $\dim V(\ker\Omega_L) = 0$;
(ii) Type II: if $\dim\ker\Omega_L = 2\dim V(\ker\Omega_L) + 1 \neq 1$;
(iii) Type III: if $\dim\ker\Omega_L < 2\dim V(\ker\Omega_L) + 1$.

4. Reduction of non-autonomous Lagrangian systems

Let L be a Lagrangian on $TQ \times \mathbb{R}$. The program of reduction we are going to develop consists of studying the structure of the quotient space $TQ \times \mathbb{R}/K$. We identify K with the gauge degrees of freedom of the system associated to the Lagrangian function, and consequently they should be removed. Supporting this physical argument, notice that if $(\Omega_L, \mathrm{d}t)$ define a precosymplectic structure on $TQ \times \mathbb{R}$, then it is well known that the quotient space $TQ \times \mathbb{R}/K$ inherits a cosymplectic structure (Ω, η), hence a dynamical vector field, its Reeb field. Both, Ω_L and $\mathrm{d}t$, are projectable along K because clearly Ω_L satisfies $\mathcal{L}_Z\Omega_L = 0$, $i_Z\Omega_L = 0$ for all $Z \in K$ and $i_Z \mathrm{d}t = 0$ (and then $\mathcal{L}_Z \mathrm{d}t = 0$) for all $Z \in K$. Also, each vector field X such that $i_X\Omega_L = 0$, $i_X \mathrm{d}t = 1$ projects onto $TQ \times \mathbb{R}/K$. Namely

$$i_{[X,Z]}\Omega_L = \mathcal{L}_X i_Z \Omega_L - i_Z(\mathcal{L}_X\Omega_L) = \qquad \forall Z \in \ker \Omega_L$$

implies that $[X, Z] \in \ker \Omega_L$, $\forall Z \in \ker \Omega_L$. Moreover, $[X, Z] \in \ker \mathrm{d}t$, $\forall Z \in \ker \mathrm{d}t$, since $\mathrm{d}t([X, Z]) = 0$. Therefore, $[X, Z] \in K$, $\forall Z \in K$. Then the Reeb vector field Γ on $TQ \times \mathbb{R}/K$ will satisfy the dynamical equations

$$i_\Gamma \Omega = 0 \qquad i_\Gamma \eta = 1 \tag{13}$$

and $\pi^*\Omega = \Omega_L$, $\pi^*\eta = \mathrm{d}t$ by construction. The notion of a Lagrangian system presupposes, however, the existence of an integrable, almost s-tangent structure on $TQ \times \mathbb{R}/K$, or the analogue of the geometrical objects $\tilde{S}$ and Δ. Therefore, if we want to obtain as a final product of the reduction procedure a regular Lagrangian system, we must find under which circumstances the adequate tensor fields pass to the quotient, and endow it with the corresponding structures, i.e. it is necessary to prove, for instance, that the triple $(\tilde{S}, \mathrm{d}t, \partial/\partial t)$ is projectable to $TQ \times \mathbb{R}/K$ and that its projection defines an integrable almost s-tangent structure on $TQ \times \mathbb{R}/K$.

The conditions of projectability under an integrable distribution for forms and vector fields are well known and have been used above. The projectability condition for (1, 1) tensor fields can be stated as follows. Let D be an integrable distribution on a manifold M such that the foliation defined by D is a fibration and let M/D be the leaf space. A tensor field R of type (1,1) on M projects onto M/D if $R(D) \subset D$ and $\mathrm{Im}(\mathcal{L}_Z R) \subset D$, $\forall Z \in D$ (see [4] for a proof). Then

Lemma 5. Let L be a degenerate Lagrangian on $TQ \times \mathbb{R}$. Then

(i) $\tilde{S}$ projects onto $TQ \times \mathbb{R}/K$ if $\mathrm{Im}(\mathcal{L}_Z\tilde{S}) \subset K$, $\forall Z \in K$.

(ii) $\partial/\partial t$ projects onto $TQ \times \mathbb{R}/K$ if $[\partial/\partial t, Z] \in K$, $\forall Z \in K$.

(iii) If $\tilde{S}$ and $\partial/\partial t$ are projectable, then S is projectable.

Proof.

(i) From lemma 3 we have $\tilde{S}(K) \subset K$.

(ii) We can write $S = \tilde{S} - (\partial/\partial t \odot \mathrm{d}t)$. Then because $\tilde{S}$, $\partial/\partial t$ and $\mathrm{d}t$ are projectable, S is projectable. □

Proposition 1.

If there exists a SODE Γ such that $i_\Gamma\Omega_L = 0$, $i_\Gamma \mathrm{d}t = 1$ and S is projectable, then the Liouville vector field Δ on $TQ \times \mathbb{R}$ and $\tilde{S}$ are both projectable.

Proof. (i) Let Z be an arbitrary vector field in K. Then using $S(\Gamma) = \Delta$ we have

$$[Z, \Delta] = (\mathcal{L}_Z S)\Gamma + S[Z, \Gamma] .$$

We know that $[Z, \Gamma] \in K$ and $S(K) \in K$, then $S[Z, \Gamma] \in K$. Also, because S is projectable, $\mathrm{Im}(\mathcal{L}_Z S) \in K \Rightarrow (\mathcal{L}_Z S)\Gamma \in K$. Then $[Z, \Delta] \in K$, $\forall Z \in K \Rightarrow \Delta$ is projectable.

(ii) $\tilde{S} = S - \Delta \otimes \mathrm{d}t$. Since S, Δ and $\mathrm{d}t$ are projectable, then $\tilde{S}$ is projectable.□

4.1. Type II Lagrangians

Even if a degenerate Lagrangian L is such that the projectability conditions (i), (ii) in lemma 5 are satisfied, the quotient structure $(\bar{F}, \tau, T)$, where $\bar{F}$ and T are the projections of $\tilde{S}$ and $\partial/\partial t$ respectively, is not necessarily an integrable almost s-tangent structure on $TQ \times \mathbb{R}/K$. It could happen that the rank of the projected tensor $\bar{F}$ is wrong. The following proposition shows that type II Lagrangians are the only ones that could give almost s-tangent structures on the quotient space.

Proposition 2. Let L be a degenerate Lagrangian such that the tensors $\tilde{S}$ and $\partial/\partial t$ are projectable under K. Then the projected tensors define an integrable almost s-tangent structure if L is of type II.

Proof. Let us assume first that the tensors $\tilde{S}$ and $\partial/\partial t$ are projectable. By definition of $\bar{F}$, for any $m \in TQ \times \mathbb{R}/K$ and $\tilde{\xi} \in T_m(TQ \times \mathbb{R}/K)$, we have $\bar{F}_m(\tilde{\xi}) = \pi_*(S_y(\xi))$ for any $y \in \pi^{-1}\{m\}$ and any $\xi \in T_y(TQ)$ such that $\pi_*(\xi) = \tilde{\xi}$. Therefore, $\dim \bar{F}_m(T_m(TQ \times \mathbb{R}/K)) = \dim \pi_*(\tilde{S}_y(T_y(TQ \times \mathbb{R})))$. From lemma 1 we know that $\mathrm{Im}\,\tilde{S} = V(TQ \times \mathbb{R}) \oplus T\mathbb{R}$. Then, it is easy to see that $\dim \pi_*(\tilde{S}_y(T_y(TQ \times \mathbb{R}))) = \dim \tilde{S}_y(T_y(TQ \times \mathbb{R})) - \dim \ker(\pi)_*(\tilde{S}_y(T_y(TQ \times \mathbb{R}))) = \dim V_y(TQ \times \mathbb{R}))) + 1 - \dim V_y(K) = \frac{1}{2}\dim TQ - \dim V(K) + 1$. But because $(\bar{F}, \tau, T)$ defines an integrable s-tangent structure on $TQ \times \mathbb{R}/K$, rank $\bar{F} = \dim \bar{F}_m(T_m(TQ \times \mathbb{R}/K)) = \frac{1}{2}(\dim TQ - \dim K) + 1$. Then $\dim V(K) = \frac{1}{2}\dim K$, and L is a type II Lagrangian.

Conversely, if L is of type II, $\dim V(K) = \frac{1}{2}\dim K$, then, with the argument used before, rank$\bar{F} = \dim \bar{F}_m(T_m(TQ \times \mathbb{R}/K)) = \frac{1}{2}\dim TQ - \dim V(K) + 1$. This implies immediately rank $\bar{F} = \frac{1}{2}\dim(TQ \times \mathbb{R}/K) + 1$. The other conditions in the definition of an almost s-tangent structure are immediately verified: $\bar{F}^2 = T \otimes \tau$, $i_T\tau = 1$, $N_{\bar{F}} = 0$ and τ is closed. □

Even if the integrable almost s-tangent structure on $TQ \times \mathbb{R}$ is projectable, this does not imply that the Lagrangian function itself will project to the reduced space. There will exist a local Lagrangian function $\tilde{L}_U : U \subset TQ \times \mathbb{R}/K \to \mathbb{R}$ defining locally the Lagrangian structure of the SODE Hamiltonian cosymplectic system Γ on the quotient (13), but L and L_U (where $L_U = \tilde{L}_U \circ \pi$) will be gauge-equivalent; there will exist a closed 1-form α_U on $\tilde{\tau}_Q(\pi^{-1}(U)) \subset Q \times \mathbb{R}$ such that $L = L_U + \hat{\alpha}_U$ up to a constant. This family of 1-forms α_U defines an obstruction to the existence of a globally defined projectable Lagrangian $\tilde{L}$ on $TQ \times \mathbb{R}/K$ which has as dynamics the projected equations of motion (13).

The following proposition gives several characterizations of type II Lagrangian systems.

Proposition 3. The Lagrangian L is of type II if $\bar{S}(K) = \tilde{S}(K) = S(K) = V(K)$.

Proof. Because of lemma 3 we have that $S(K) \subset V(K)$. On the other hand because of (10) and lemma 4, we have that $\dim S(K) = \dim K - \dim V(K)$. For type II Lagrangian functions $\dim V(K) = \frac{1}{2} \dim K$. Then $\dim S(K) = \dim V(K)$ and the result follows. □

Corollary 1. If L is a type II Lagrangian function, then $V(\ker \Omega_L) \subset S(\ker \Omega_L)$ and $\dim S(\ker \Omega_L) - \dim(\ker \Omega_L \cap \operatorname{Im} S) \leqslant 1$.

Proof. $\ker \Omega_L \cap \operatorname{Im} S = K \cap \operatorname{Im} S = S(K) \subset S(\ker \Omega_L)$. Now, we observe that $\dim S(\ker \Omega_L) \leqslant \dim S(K) + 1$. □

One of the main properties of type II Lagrangians is given by the following proposition.

Proposition 4. If L is a type II Lagrangian function, then there exists a projectable SODE Γ such that $i_\Gamma \Omega_L = 0$, $i_\Gamma \mathrm{d}t = 1$.

Proof. We know that there exists a vector field X such that $i_X \Omega_L = 0$, $i_X \mathrm{d}t = 1$. Now, we want to construct a vector field Γ satisfying $S(\Gamma) = \Delta$ and the equations of motion $i_\Gamma \Omega_L = 0$, $i_\Gamma \mathrm{d}t = 1$. From (7) we get that $\bar{S}(X)\Omega_L = 0$, but $\bar{S}(X) = S(X) - \Delta$ because $i_X \mathrm{d}t = 1$. Then $S(X) - \Delta \in K$ and also $S(X) - \Delta \in \operatorname{Im} S$, but because L is a type II Lagrangian function, $S(K) = V(K)$, and then there exists a vector field $Y \in K$ such that $S(Y) = S(X) - \Delta$. If we set $\Gamma = X - Y$, then we have $S(\Gamma) = \Delta$, $i_\Gamma \mathrm{d}t = 1$ and $i_\Gamma \Omega_L = 0$. Finally, because Γ satisfies the equations of motion, we have already proved that Γ is projectable onto $TQ \times \mathbb{R}/K$. □

Corollary 2. If L is a type II Lagrangian and S is projectable, then the Liouville vector field Δ on $TQ \times \mathbb{R}$ and $\tilde{S}$ are also projectable.

Proof. The proof follows from propositions 1 and 2. □

Example. We are going to apply the results obtained before to the special class of Lagrangians discussed in [2]. There, de León *et al* studied degenerate non-autonomous Lagrangians satisfying the conditions

$$\bar{S}(\ker \Omega_L) = \ker \Omega_L \cap \operatorname{Im} \bar{S} \tag{14}$$

$$S(\ker \Omega_L) = \ker \Omega_L \cap \operatorname{Im} S\,. \tag{15}$$

It is clear from the condition (14) that $\bar{S}(K) = K \cap V(TQ \times \mathbb{R})$, i.e. $S(K) = V(K)$. In fact, we let X be a vector on $V(K)$. This means that X is in $\ker \Omega_L \cap \ker \mathrm{d}t \cap V(TQ \times \mathbb{R} \subset \ker \Omega_L \cap \operatorname{Im} \bar{S}$, and hence there exists $U \in \ker \Omega_L$ such that $\bar{S}(U) = X$. If U does not belong to $\ker \mathrm{d}t$, this implies that $\mathrm{d}t(U) \neq 0$, then $\mathrm{d}t(\bar{S}(U) \neq 0$, which is a contradiction. Therefore, condition (14) implies that these Lagrangians are particular examples of type II Lagrangians. Besides, the assumption $\bar{S}(\ker \Omega_L) = \ker \Omega_L \cap \operatorname{Im} \bar{S}$ is stronger than $S(K) = V(K)$. For instance if Γ is a vector field in $\ker \Omega_L$ such that $i_\Gamma \mathrm{d}t \neq 0$, it is possible that $i_{\bar{S}(\Gamma)} \Omega_L \neq 0$.

Condition (15) also implies by itself that the Lagrangian function is of type II. This can be seen by an argument similar to the argument used previously for condition (14). In addition, because proposition 4 assures that there exists a SODE Γ such that $i_\Gamma \Omega_L = 0$, $i_\Gamma \mathrm{d}t = 1$, and as a consequence of proposition 3 we have that condition (15) implies that $S(\ker \Omega_L) = \ker \Omega_L \cap \mathrm{Im}\, S = S(K) \subset K$. In particular, $S(\Gamma) \in K$, but $S(\Gamma) = \Delta \Rightarrow \Delta \in K$. This means that the Liouville vector field will be projectable with projection equal to zero.

4.2. *Reduction of a class of type III Lagrangians*

In this section we would like to study some aspects of the reduction of type III non-autonomous Lagrangian systems, $\dim K < 2 \dim V(K)$. Because of the inequality $\dim V(K) \leqslant \dim K \leqslant 2 \dim V(K)$, the most degenerate case would happen if the Lagrangian is such that $\dim K = \dim V(K)$, that is, $\ker \Omega_L$ consists only of a vector field solution of the dynamical equations and vertical vector fields. Lagrangians satisfying this condition will be called *completely degenerate* Lagrangians and have been already studied in the autonomous case [7] to solve the inverse problem of mechanics for certain coupled differential equations.

4.2.1. Completely degenerate Lagrangians. Let L be a Lagrangian such that $\dim V(K) = \dim K$. Then necessarily $K = V(K)$ and K is an integrable distribution consisting only of vertical fields.

A model for completely degenerate Lagrangians. We will now discuss the main model of type III Lagrangians satisfying $\dim V(K) = \dim K$. The configuration space is going to be the cartesian product $M \times N \times \mathbb{R}$, with x^i the local coordinates on M and z^α the local coordinates on N. Consider the Lagrangian function

$$L = L_0(x, \dot{x}; z, t) + A_\alpha(x, \dot{x}; z, t)\dot{z}^\alpha . \tag{16}$$

The Cartan 1-form is

$$\Theta_L = \left(\frac{\partial L_0}{\partial \dot{x}^i} + \dot{z}^\alpha \frac{\partial A_\alpha}{\partial \dot{x}^i}\right) \mathrm{d}x^i + A_\alpha \mathrm{d}z^\alpha + \left(L_0 - \dot{x}^i \frac{\partial L_0}{\partial \dot{x}^i} - \dot{x}^i \dot{z}^\alpha \frac{\partial A_\alpha}{\partial \dot{x}^i}\right) \mathrm{d}t \tag{17}$$

and the Cartan 2-form,

$$\begin{aligned}
\Omega_L = {} & \left(\frac{\partial^2 L_0}{\partial \dot{x}^i \partial \dot{x}^j} + \frac{\partial^2 A_\alpha}{\partial \dot{x}^i \partial \dot{x}^j} \dot{z}^\alpha\right) \mathrm{d}\dot{x}^j \wedge \mathrm{d}x^i + \left(\frac{\partial^2 L_0}{\partial \dot{x}^i \partial x^j} + \frac{\partial^2 A_\alpha}{\partial \dot{x}^i \partial x^j} \dot{z}^\alpha\right) \mathrm{d}x^j \wedge \mathrm{d}x^i \\
& + \left(\frac{\partial^2 L_0}{\partial \dot{x}^i \partial z^\alpha} + \frac{\partial^2 A_\beta}{\partial \dot{x}^i \partial z^\alpha} \dot{z}^\beta - \frac{\partial A_\alpha}{\partial x^i}\right) \mathrm{d}z^\alpha \wedge \mathrm{d}x^i \\
& + \frac{\partial A_\alpha}{\partial \dot{x}^i} \mathrm{d}\dot{x}^i \wedge \mathrm{d}z^\alpha + \frac{\partial A_\alpha}{\partial \dot{x}^i} \mathrm{d}\dot{z}^\alpha \wedge \mathrm{d}x^i + F_{\alpha\beta} \mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta \\
& + \left(-\frac{\partial L_0}{\partial x^i} + \frac{\partial^2 L_0}{\partial \dot{x}^i \partial t} + \frac{\partial^2 A_\alpha}{\partial \dot{x}^i \partial t} \dot{z}^\alpha + \dot{x}^j \frac{\partial^2 L_0}{\partial x^i \partial \dot{x}^j}\right. \\
& \left. + \dot{z}^\alpha \dot{x}^j \frac{\partial^2 A_\alpha}{\partial x^i \partial \dot{x}^j}\right) \mathrm{d}t \wedge \mathrm{d}x^i
\end{aligned}$$

$$+\left(\dot z^\alpha \dot x^j \frac{\partial^2 A_\alpha}{\partial \dot x^i \partial \dot x^j} + \dot x^j \frac{\partial^2 L_0}{\partial \dot x^i \partial \dot x^j} + \frac{\partial A_\alpha}{\partial \dot x^i}\dot z^\alpha\right) \mathrm{d}t \wedge \mathrm{d}\dot x^i$$

$$+\left(-\frac{\partial L_0}{\partial z^\alpha} + \frac{\partial A_\alpha}{\partial t} + \dot z^\beta \dot x^i \frac{\partial^2 A_\beta}{\partial \dot x^i \partial z^\alpha} + \dot x^i \frac{\partial^2 L_0}{\partial z^\alpha \partial \dot x^i}\right) \mathrm{d}t \wedge \mathrm{d}z^\alpha$$

$$+\dot x^i \frac{\partial A_\alpha}{\partial \dot x^i} \mathrm{d}t \wedge \mathrm{d}\dot z^\alpha \,. \tag{18}$$

The matrix associated with Ω_L in the basis of 1-forms $(\mathrm{d}x^i, \mathrm{d}\dot x^i; \mathrm{d}z^\alpha, \mathrm{d}\dot z^\alpha; \mathrm{d}t)$ is

$$(\Omega_L) = \begin{pmatrix} M & W & R & -T & H \\ -W^t & 0 & T & 0 & K \\ -R^t & -T^t & F & 0 & V \\ T^t & 0 & 0 & 0 & Z \\ -H^t & -K^t & -V^t & -Z^t & 0 \end{pmatrix} \tag{19}$$

where

$$M_{ij} = \frac{\partial^2 L_0}{\partial \dot x^i \partial x^j} - \frac{\partial^2 L_0}{\partial \dot x^j \partial x^i} + \left(\frac{\partial^2 A_\alpha}{\partial \dot x^i \partial x^j}\dot z^\alpha - \frac{\partial^2 A_\alpha}{\partial \dot x^j \partial x^i}\dot z^\alpha\right) \tag{20}$$

$$W_{ij} = \frac{\partial^2 L_0}{\partial \dot x^i \partial \dot x^j} + \frac{\partial^2 A_\alpha}{\partial \dot x^i \partial \dot x^j}\dot z^\alpha \tag{21}$$

$$R_{i\alpha} = \frac{-\partial A_\alpha}{\partial x^i} + \frac{\partial^2 L_0}{\partial \dot x^i \partial z^\alpha} + \frac{\partial^2 A_\beta}{\partial \dot x^i \partial z^\alpha}\dot z^\beta \tag{22}$$

$$T_{i\alpha} = -\frac{\partial A_\alpha}{\partial \dot x^i} \tag{23}$$

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial z^\alpha} - \frac{\partial A_\alpha}{\partial z^\beta} \tag{24}$$

$$Z_\alpha = \dot x^i \frac{\partial A_\alpha}{\partial \dot x^i} \tag{25}$$

$$H_i = -\frac{\partial L}{\partial x^i} + \frac{\partial^2 L_0}{\partial \dot x^i \partial t} + \frac{\partial^2 A_\alpha}{\partial \dot x^i \partial t}\dot z^\alpha + \dot x^j \frac{\partial^2 L_0}{\partial x^i \partial \dot x^j} + \frac{\partial A_\alpha}{\partial x^i}\dot z^\alpha + \dot x^j \frac{\partial^2 A_\alpha}{\partial x^i \partial \dot x^j} \tag{26}$$

$$K_i = \dot x^j \frac{\partial^2 A_\alpha}{\partial \dot x^i \partial \dot x^j}\dot z^\alpha + \dot x^j \frac{\partial^2 L_0}{\partial \dot x^i \partial \dot x^j} + \frac{\partial A_\alpha}{\partial \dot x^i}\dot z^\alpha \tag{27}$$

$$V_\alpha = \frac{\partial L_0}{\partial z^\alpha} + \frac{\partial A_\alpha}{\partial t} + \dot x^i \frac{\partial^2 A_\beta}{\partial \dot x^i \partial z^\alpha}\dot z^\beta + \dot x^i \frac{\partial^2 L_0}{\partial z^\alpha \partial \dot x^i} \,. \tag{28}$$

We must observe that the logical process is to reduce the total space $T(M \times N) \times \mathbb{R}$ by the vertical vector fields in TN to obtain the reduced space $TM \times N \times \mathbf{R}$. Then it is necessary that the terms in Ω_L with the form $\mathrm{d}\dot z^\alpha \wedge (\ldots)$ will vanish. We can reduce Ω_L if and only if the submatrices T and Z are zero. However, it is clear from (23) and (25) that $T = 0$ implies $Z = 0$. Therefore, we only need to impose that $\partial A_\alpha / \partial \dot x^i = 0$. If we project Ω_L onto $TM \times N \times \mathbb{R}$ we obtain the 2-form $\tilde\Omega_L$ with associated matrix

$$(\tilde\Omega_L) = \begin{pmatrix} M & W & R & H \\ -W^t & 0 & 0 & K \\ -R^t & 0 & F & N \\ -H^t & -K^t & -N^t & 0 \end{pmatrix} \tag{29}$$

that we will assume of maximal rank. Notice that this implies that the dimension of N is even. It is clear that under these circumstances the gauge distribution K of the Lagrangian

$$L = L_0(x, \dot{x}; z, t) + A_\alpha(x; z, t)\dot{z}^\alpha$$

is $K = V(TN)$. The equations of motion are the solutions of the dynamical equations

$$i_\Gamma \tilde{\Omega}_L = 0 \qquad i_\Gamma \mathrm{d}t = 1 \tag{30}$$

and if the vector field Γ is written in local coordinates as

$$\Gamma = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + f^i \frac{\partial}{\partial \dot{x}^i} + g^\alpha \frac{\partial}{\partial z^\alpha}$$

the generalized forces f^i and g^α are given by the expressions

$$\begin{aligned} f &= W^{-1} R F^{-1}(N - R^t \dot{x}) - W^{-1} M \dot{x} - W^{-1} H \\ g &= F^{-1}(R^t \dot{x} - V)\,. \end{aligned}$$

Projectability of the geometric structures. Let $Z = B^i \partial/\partial \dot{q}^i$ be a vector field in $K = V(K)$. The tensor field $\bar{S}$ projects onto $TQ \times \mathbb{R}/K$ if and only if $\mathrm{Im}(\mathcal{L}_Z \bar{S}) \subset K$ for any $Z \in K$ (see lemma 5). In local coordinates,

$$\mathcal{L}_Z \bar{S} = -\frac{\partial B^i}{\partial t}\frac{\partial}{\partial \dot{q}^i} \otimes \mathrm{d}t - \frac{\partial B^j}{\partial \dot{q}^i}\frac{\partial}{\partial \dot{q}^j} \otimes \mathrm{d}q^i\,. \tag{31}$$

Then it is clear that $\mathrm{Im}\,\mathcal{L}_Z \bar{S}$ is spanned by vertical vector fields with coefficients $\partial B^i/\partial t$ and $\partial B^j/\partial \dot{q}^i$. On the other hand, $\partial/\partial t$ projects onto $TQ \times \mathbb{R}/K$ if and only if $[\partial/\partial t, Z] \in K, \forall Z \in K$. This means that

$$\left[\frac{\partial}{\partial t}, Z\right] = \frac{\partial B^i}{\partial t}\frac{\partial}{\partial \dot{q}^i} \in K\,. \tag{32}$$

Therefore, we have the following result.

Proposition 5. Let L be a completely degenerate Lagrangian. Then
 (i) $\bar{S}$ projects onto $TQ \times \mathbb{R}/K$ if

$$(\partial B^i/\partial t)(\partial^2 L \partial \dot{q}^i \partial \dot{q}^j) = (\partial B^j/\partial \dot{q}^i)(\partial^2 L \partial \dot{q}^j \partial \dot{q}^k) = 0$$

for all $Z = B^i \partial/\partial \dot{q}^i \in K$.
 (ii) $\partial/\partial t$ projects onto $TQ \times \mathbb{R}/K$ if $(\partial B^i/\partial t)(\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j) = 0$ for Z as before.

Corollary 3. If $\bar{S}$ is projectable onto $TQ \times \mathbb{R}/K$ then S is projectable.

Proof. From proposition 5 we get that if $\bar{S}$ is projectable then $\partial/\partial t$ is projectable, and because $\mathrm{d}t$ is always projectable, we get that S is projectable. $\square$

We would like to know something about the projectability of $\tilde{S}$ and Δ. First of all, it is clear that if S and Δ are projectable, then $\tilde{S}$ is projectable. On the other hand Δ is projectable if and only if $[\Delta, Z] \in K$, for any $Z \in K$. But locally,

$$[\Delta, Z] = \left(\dot{q}^i \frac{\partial B^j}{\partial \dot{q}^i} - B^j\right) \frac{\partial}{\partial \dot{q}^j} . \tag{33}$$

From proposition 5 we finally obtain

Corollary 4. If $\bar{S}$ is projectable onto $TQ \times \mathbb{R}/K$ then the Liouville vector field Δ is projectable.

Therefore, what we have shown is that the projectability of $\bar{S}$ implies the projectability of all the other structures. In particular, $\bar{S}$ is always projectable if the distribution K is time-independent and constant along the fibres of $TQ \times \mathbb{R}$, i.e. if the distribution K can be generated locally by a family of vector fields $Z_1, \ldots Z_k$ such that if $Z_j = B_j^i \partial/\partial \dot{q}^i$, then $\partial B_j^i/\partial t = \partial B_j^i/\partial \dot{q}^l = 0$. Then, the conditions of proposition 5 are automatically satisfied. It is easy to show that the previous conditions are equivalent to the existence of distribution F on Q such that $K = F^V$, where F^V denotes the vertical lifting of F, i.e. if $X = X^i \partial/\partial q^i$ is a vector field in F, then $X^V = X^i \partial/\partial \dot{q}^i$.

Theorem 2. Let L be a completely degenerate Lagrangian such that K is the vertical lifting of an integrable distribution F on Q, then the tensors $\bar{S}$, $\tilde{S}$, S, Δ, $\partial/\partial t$ all project to $TQ \times \mathbb{R}/K$ which locally has the form $TM \times N \times \mathbb{R}$.

Proof. It is clear that all the tensors and vector fields project because of propositions 5 and corollaries 3 and 4. The local structure of the quotient space can be obtained from the following considerations. The vertical bundle $V(TQ \times \mathbb{R})$ can be identified with the Whitney sum of two copies of the vector bundle $J^1(\mathbb{R}, Q)$ over $Q \times \mathbb{R}$. In general, if $E \to M$ is a vector bundle over M, the vertical vector bundle $V(E)$ is isomorphic to $E \oplus E$, the identification provided by the bundle map $(\xi, \zeta) \in E \oplus E \mapsto \mathrm{d}/\mathrm{d}t(\xi + t\zeta)\,|_{t=0} \in T_\xi E$. In this sense the distribution K, being the vertical lifting of F, can be identified with a sub-bundle F of $TQ \times \mathbb{R} \to Q \times \mathbb{R}$. The quotient manifold Q/F will be denoted by M; then it is easy to show that the total quotient manifold $TQ \times \mathbb{R}/K$ is locally diffeomorphic to $TM \times N \times \mathbb{R}$ where N is a leaf of the integrable distribution F. Let m be a point in M, i.e. a leaf of the integrable distribution F in Q. We will denote this leaf by N; then, because of the fibration property of K, hence of F, there exists an open neighbourhood U of m such that $\pi^{-1}(U) \cong U \times N \times \mathbb{R}$. Locally, $Q \times \mathbb{R}$ can be factorized as $M \times N \times \mathbb{R}$ and the distribution K is precisely the tangent bundle of N in this factorization. □

4.2.2. $\dim V(K) < \dim K$. For general type III Lagrangians is impossible to go beyond the generic result on projectability of the natural tensors in $TQ \times \mathbb{R}$ unless we make some specific assumptions on the structure of K. In particular there is an important family of Type III Lagrangians extending the particular case of Lagrangians such that K is a tangent algebra considered in [4] and the Lagrangians such that K is an s-tangent distribution considered in [2].

Definition 1. A distribution D on $TQ \times \mathbb{R}$ will be said to be almost tangent if there exist two integrable distributions $E \subset F$ on $Q \times \mathbb{R}$ locally generated by families of vector fields $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_l$ respectively, such that D is locally generated by the family of vector fields $X_1^{(1)}, \ldots, X_k^{(1)}; Y_1^V, \ldots, Y_l^V$, where $X^{(1)}$ denotes the first extension of the vector field X and Y^V denotes the vertical lifting of Y.

We will denote the almost tangent distribution D as $E^{(1)} \odot F^V$. It is clear that if $E = F$, the distribution D is an s-tangent distribution as in [2] and if E is time-independent, D is simply TE with the notation in [4]. It is also evident from the definition that if a degenerate Lagrangian has gauge distribution K of almost tangent type $E^{(1)} \odot F^V$, then the Lagrangian L is a projectable of type III with $\dim V(K) = \dim F < \dim F + \dim E = \dim K$.

Corollary 5. Let L be a type III Lagrangian function such that $K = E^{(1)} \odot F^V$ is almost tangent and time-independent, then the tensors $\bar{S}$, $\tilde{S}$, S, Δ, $\partial/\partial t$ all project to $TQ \times \mathbb{R}/K$ which has locally the form $TM \times N \times \mathbb{R}$, where M is transverse to a leaf of F and N is the quotient of a leaf of F by E.

Proof. Applying theorem 2, the quotient space corresponding to the gauge distribution $K = F^V$ is $TM \times P \times \mathbb{R}$, where P is a leaf of F and M is transverse to the foliation F. But the gauge distribution still contains E; then E defines a foliation of F, hence the quotient space has the form $TM \times N \times \mathbb{R}$ where $N = P/E$. □

Linear Lagrangians on Lie groups. The most significant example of this situation happens when we consider linear Lagrangians in groups defined by left-invariant 1-forms. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We will identify the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ with the space of left-invariant 1-forms on G. In other words if $\mu \in T_e^* G$ then there exists a unique left-invariant 1-form α_μ such that $\alpha_\mu(e) = \mu$, defined by $\alpha_\mu(g) = L_g^* \mu$. Then we can consider the Lagrangian function on $TG \times \mathbb{R}$

$$L_{\mu,V}(g, \dot{g}, t) = \langle \alpha_\mu(g), \dot{g} \rangle - V(g, t) \tag{34}$$

with V a G-invariant function on $G \times \mathbb{R}$. It is easy to check that the Cartan 1-form $\Theta_{\mu,V}$ of $L_{\mu,V}$ is given by

$$\Theta_{\mu,V} = \tau_G^* \alpha_\mu - V \mathrm{d}t \tag{35}$$

and the Cartan 2-form $\Omega_{\mu,V} = \mathrm{d}\Theta_{\mu,V}$ is given by

$$\Omega_{\mu,V} = \tau_G^* \mathrm{d}\alpha_\mu - \mathrm{d}V \wedge \mathrm{d}t\,. \tag{36}$$

Proposition 6. In the conditions above the gauge distribution K of $L_{\mu,V}$ is $\mathfrak{g}_\mu \odot \mathfrak{g}$, the quotient space $TG \times \mathbb{R}/K$ is isomorphic to $\mathcal{O}_\mu \times \mathbb{R}$ where $\mathcal{O}_\mu$ denotes the coajoint orbit of G passing through μ, and the reduced dynamics is the cosymplectic Hamiltonian system defined by the projected function V.

Proof. From equation (36), we easily get that $V(TG \times \mathbb{R}) = \mathfrak{g}$ is contained in $\ker \Omega_{\mu,V}$, and this is just the vertical part. Using the left decomposition of $TG = G \times \mathfrak{g}$, the horizontal vector Z fields on K, satisfying $i_Z \mathrm{d}\alpha_\mu = 0$. It suffices to consider left-invariant vector fields. Then, the previous condition is equivalent to the equation in the tangent space at the identity element

$$\mu([Z(e), \xi]) = 0 \qquad \forall \xi \in \mathfrak{g} \tag{37}$$

in other words, $Z \in K$ if $Z(e) \in \mathfrak{g}_\mu$, the isotropy algebra of μ. □

Remarks. (i) This theorem is the Lagrangian counterpart of the Kostant–Kirillov–Souriau theorem and gives an alternative description of coadjoint orbits in terms of tangent bundle geometry.

(ii) If $G_\mu = G$, or equivalently if μ is a 1-cocycle in Chevalley cohomology, then L_μ is a zero-gauge equivalent Lagrangian because $\mathrm{d}\alpha_\mu = 0$. That means that $\Omega_\mu = 0$ and $TG \times \mathbb{R}/K = \mathbb{R}$.

(iii) A particular case of this result has been used recently to discuss the quantization and structure of gauge theories in coadjoint orbits (see for instance [1] and [9]).

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